Numerical Evaluation of limits

Note Title 17/09/2008

The material for this lecture is taken from section 2.2, with a little look-ahead to some important upcoming limits.

<u>Some</u> <u>preliminary</u> <u>remarks</u>

- $\lim_{x \to a} f(x)$ will be a common thing for you to <u>do</u>.

- Most of the time you will use <u>continuity</u> (Section 2.5)

$$\lim_{x \to a} f(x) = f\left(\lim_{x \to a} x\right) = f(a)$$

<u>if</u> $f(x)$ is continuous (cts) at $x = a$

- For some basic limits we will have to use the definition (Section 2.4: not covered) and geometry and inequalities.
- In lieu of the definition, numerical experiments, when done carefully, can evaluate several difficult cases.

- The book is <u>too</u> <u>cautious</u> about calculators/computers (see pp 1–9, and section 1.4) – we can really get the right answers, if we are aware of one simple thing:

Floating-Point

Addition is not the same as real!

Example: In 5 digit arithmetic

$$1.0000 + 0.1234 - 1.0000 = 0.1234$$

$$1.0000 + 0.01234 - 1.0000 = 0.0123$$

$$1.000 + 0.001234 - 1.0000 = 0.0012$$

$$1.000 + 0.0001234 - 1.0000 = 0.0001$$

$$1.000 + 0.00001234 - 1.0000 = 0.0000$$

what happens is that

digits fall off the end in +

$$\begin{array}{l} 1.0000 \\ .001234 \\ \hline 1.0012\,\ell\,\ell \end{array}$$

ℓ = lost

because only 5 digits are kept.

This error is <u>revealed</u> when 1 is subtracted.

Fine, but who cares? Why add 1 if you're only going to subtract it again?

Example

$$\lim_{x \to 0} \frac{1 - \sqrt{1-x}}{x}$$

Oh, can't subtract 1 <u>exactly</u>

$x = 10^{-3}$, say: $1-x = 0.9990$

$\sqrt{1-x} = 0.9995$ in 4-digit arith

$$\frac{1-\sqrt{1-x}}{x} = \frac{0.0005}{0.001} = 0.5 \quad \text{ok.}$$

but $x = 5 \cdot 10^{-4}$ $\quad 1-x = 0.9995$

$\sqrt{1-x} = 0.9998$ to 4 places

$$\frac{1-\sqrt{1-x}}{x} = \frac{0.0002}{0.001} = 0.2$$

but this result is contaminated by the rounding error in $1-x$.

We fix this by using tricks to improve the numerics. We want to subtract the 1 analytically (exactly) if we can.

$$\frac{1-\sqrt{1-x}}{x} = \frac{\left(1-\sqrt{1-x}\right)}{x}\overbrace{\frac{\left(1+\sqrt{1-x}\right)}{\left(1+\sqrt{1-x}\right)}}^{\text{trick}}$$

$$= \frac{1-\left(\sqrt{1-x}\right)^2}{x\left(1+\sqrt{1-x}\right)} = \frac{1-(1-x)}{x\left(1+\sqrt{1-x}\right)}$$

$$= \frac{x}{x\left(1+\sqrt{1-x}\right)} = \frac{1}{1+\sqrt{1-x}} \qquad (x \neq 0)$$

Now no subtraction will reveal the numerical error in $1-x$.

No matter what small x we take, we get a number near $1/2$ for the limit. In fact

$$\frac{1}{1+\sqrt{1-x}} \quad \text{is continuous at } x=0$$

and so we see $\displaystyle\lim_{x \to 0} \frac{1-\sqrt{1-x}}{x} = \frac{1}{2}$.

Sometimes problems never arise.

$\displaystyle\lim_{h \to 0} \frac{\sin(h)}{h}$ take $h = 1.234 \cdot 10^{-7}$

then $\displaystyle\frac{\sin(h)}{h} = 1$

on my calculator.

$\displaystyle\lim_{h \to 0} \ln h = -\infty$ is also easy.

Sometimes problems are <u>hidden</u>

$\displaystyle\lim_{h \to 0} \frac{\ln(1+h)}{h}$ looks ok, <u>but</u>...

$h = 1.234 \cdot 10^{-6}$ $\qquad \frac{\ln(1+h)}{h} = 0.999996$

$h = 1.234 \cdot 10^{-9}$ $\qquad \frac{\ln(1+h)}{h} = 0.9966$

but it's the second that is <u>less</u> accurate (though closer to the limit point).

The cure for <u>this</u> limit is a specially accurate routine (called LNP1 on my calculator)

LNP1(h) = ln(1+h) mathematically

but it gets <u>all digits correct</u>.

(see N.J. Higham, Accuracy & Stability in Numerical Analysis).

Still, even with the inaccurate routine, we could easily guess

$$\lim_{h \to 0} \frac{\ln(1+h)}{h} = 1$$

(and it is).

We <u>wouldn't be fooled</u> by taking h too small

$$\begin{array}{r l} & 1.00000000000000 \\ + & .000000000000000123456 \\ \hline & 1.00000000000000\ell\ell\ell\ell\ell\ell\ell \end{array}$$

and then taking ln

(effectively subtracting 1, and revealing the rounding error)

Similarly

Show by numerical experiment

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

$$\lim_{h\to 0} \frac{2^h - 1}{h} = \ln 2$$

$$\lim_{h\to 0} \frac{3^h - 1}{h} = \ln 3$$

$$\lim_{t\to 0} \frac{\sqrt{t^2+9} - 3}{t^2} = \frac{1}{6}$$ (example 2, p 90)

(see the nice pictures of rounding error, p 90).

problems 17 - 24 work on these concepts.

Problems 19 and 38 are calculator/computer questions.

(there is an EXPM function on my machine that computes $e^x - 1$ accurately)

I like question 42, too!

$$\lim_{x \to 1} \frac{x^3 - 1}{\sqrt{x} - 1}$$

"remember" $x^3 = (\sqrt{x})^6$

and "remember"

$$1 + a + a^2 + \cdots + a^{n-1}$$
$$- a \cdot (1 + a + a^2 + \cdots + a^{n-2} + a^{n-1}) = 1 - a^n$$

$$= (1-a)(1 + a + a^2 + \cdots + a^{n-1})$$